Pólya, GKM and Macdonald in the centre of a quantum group

Joint works with Jakub Löwit, Anton Mellit and Shon Ngo

assisted by SandboxHQ=ChatGPT/Claude + Julia/OSCAR + Lean/Mathlib

Tamás Hausel

Institute of Science and Technology Austria

http://hausel.ist.ac.at

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Project Numina

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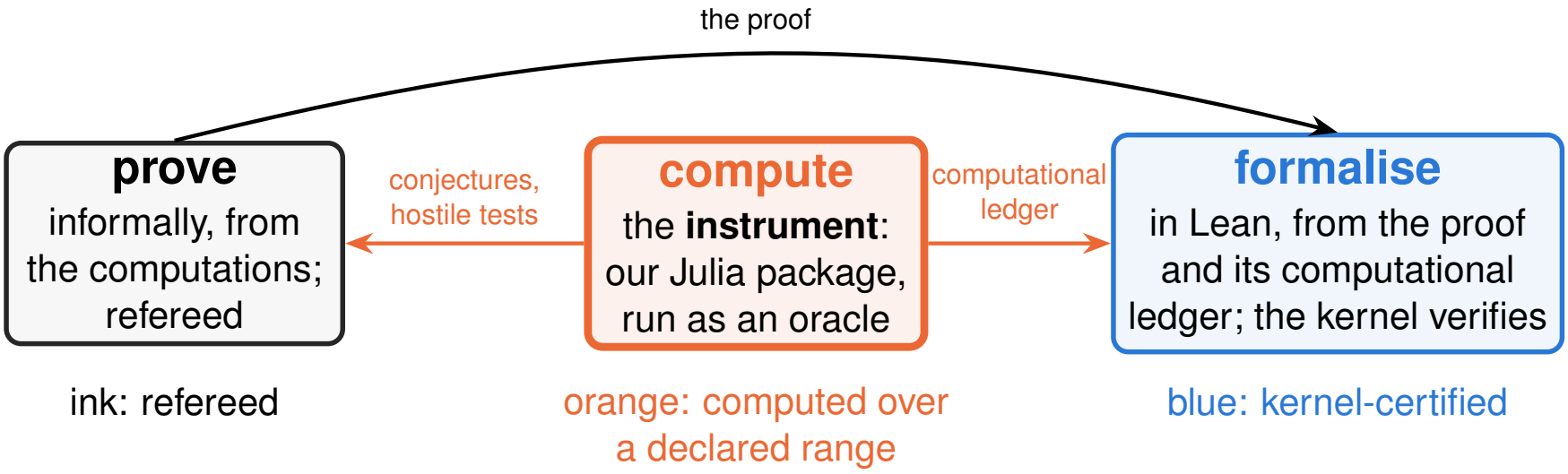
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- an almanac is a sandbox published: the GL_1 pilot,
  `https://doi.org/10.5281/zenodo.22808195`

Pólya's theorem

- Pólya (1915): asymptotics of integer-valued entire functions

Theorem (Pólya 1919)

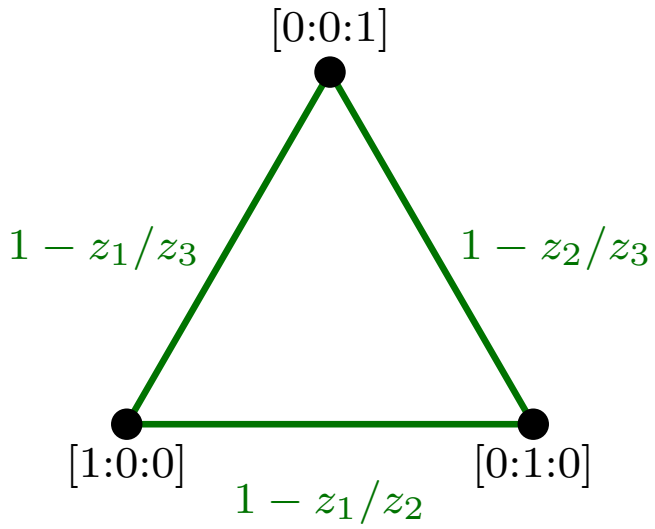
$$\{f \in \mathbb{Q}[x] : \ f(d) \in \mathbb{Z} \ \ \forall d \in \mathbb{Z}\} = \bigoplus_{r \geq 0} \mathbb{Z} \binom{x}{r}$$

- q-analogues: Gramain (1990, integer $q \geq 2$), Harman–Hopkins (2016, q an indeterminate), with basis the quantum binomial coefficients $\begin{bmatrix} d \\ r \end{bmatrix}_{q^2}$, polynomials in $Y = q^{2d}$
- $\delta_r(Y) := Y^{-\lfloor r/2 \rfloor} \begin{bmatrix} d \\ r \end{bmatrix}_{q^2}$ $(Y = q^{2d}, r \geq 0)$
- (almanac, Lean kernel)

$$\{f \in \mathbb{Q}(q)[Y^{\pm 1}] : \ f(q^{2d}) \in \mathbb{Z}[q^{\pm 1}] \ \ \forall d \in \mathbb{Z}\} = \bigoplus_{r \geq 0} \mathbb{Z}[q^{\pm 1}] \, \delta_r(Y)$$

GKM: equivariant K-theory from a graph

- $T \cong (\mathbb{C}^\times)^n \circlearrowright X$ smooth projective is *GKM*, if $|X^T| < \infty$ and $\left|\{T \cdot x \subset X : \dim T \cdot x = 1\}\right| < \infty$
- *moment graph*: vertices X^T, edges the T-invariant $\mathbb{P}^1$'s, labelled by the character χ of T on the tangent line
- (Goresky–Kottwitz–MacPherson 1998; K-theory: Knutson–Rosu 2003, Harada–Henriques–Holm 2005)



$X = \mathbb{P}^2$: the moment triangle

$$K_T(X) \cong \left\{(f_p) \in \bigoplus_{p \in X^T} R(T) : \ (1-\chi) \mid f_p - f_{p'} \ \text{ on every edge}\right\}$$

Equivariant K-theory of the affine Grassmannian

- $\mathrm{Gr} := \mathrm{GL}_n(\mathbb{C}((u)))/\mathrm{GL}_n(\mathbb{C}[[u]]) = \{\text{lattices } L \subset \mathbb{C}((u))^n\}$ the *affine Grassmannian*;
  $H := T^\vee \times \mathbb{C}^\times_{\mathrm{rot}}$, $T^\vee \subset \mathrm{GL}_n(\mathbb{C})$, $\mathbb{C}^\times_{\mathrm{rot}}$ by $u \mapsto s^2 u$, $q(s) = s$;
  fixed points $L_\lambda = \bigoplus_i u^{\lambda_i}\mathbb{C}[[u]]\varepsilon_i$, $\lambda \in \mathbb{Z}^n$
- $K_H(\mathrm{Gr})$: genuine equivariant topological K^0; GKM description (Harada–Henriques–Holm 2005):

$$K_H(\mathrm{Gr}) \cong \Big\{ (f_\lambda) \in \prod_{\lambda \in \mathbb{Z}^n} \mathbb{Z}[z_a^{\pm 1}, q^{\pm 1}] : \\ \big(1 - \tfrac{z_a}{z_b} q^{(\lambda+\nu)_a - (\lambda+\nu)_b}\big) \mid f_\lambda - f_\nu \ \text{ if } \lambda - \nu \in \mathbb{Z}(\varepsilon_a - \varepsilon_b) \Big\}$$

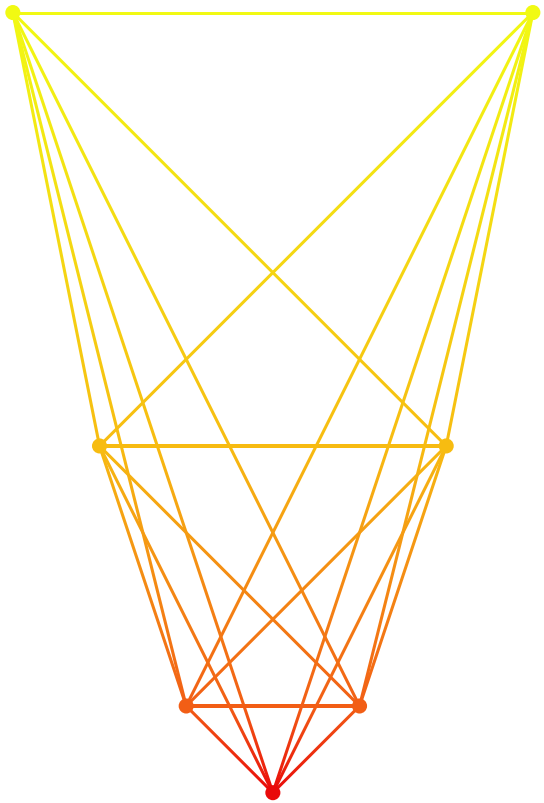
- *Bezrukavnikov classes* $b_{i,\ell}$, $1 \le i \le n$, $\ell > 0$:

$$(b_{i,\ell})_\lambda = \frac{e_i(z^\ell) - e_i\big(z^\ell q^{2\ell\lambda}\big)}{1 - q^{2\ell}}$$

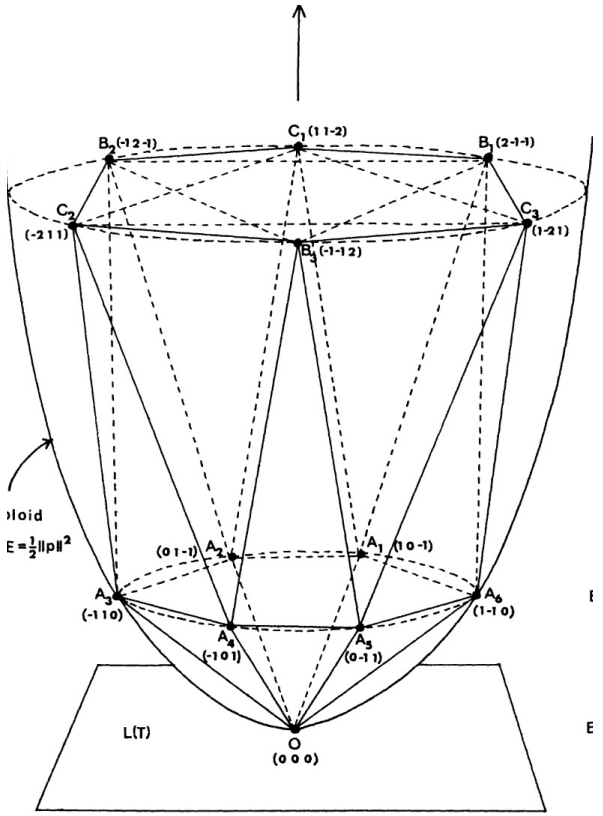
Theorem (Löwit, 2026; conjectured by Bezrukavnikov 2023)

$K_H(\mathrm{Gr}; \mathbb{C})$ is topologically generated over $K_H(\mathrm{pt}; \mathbb{C})$ by $\{b_{1,\ell}\}_{\ell \ge 1}$.

Moment polytope and GKM graph

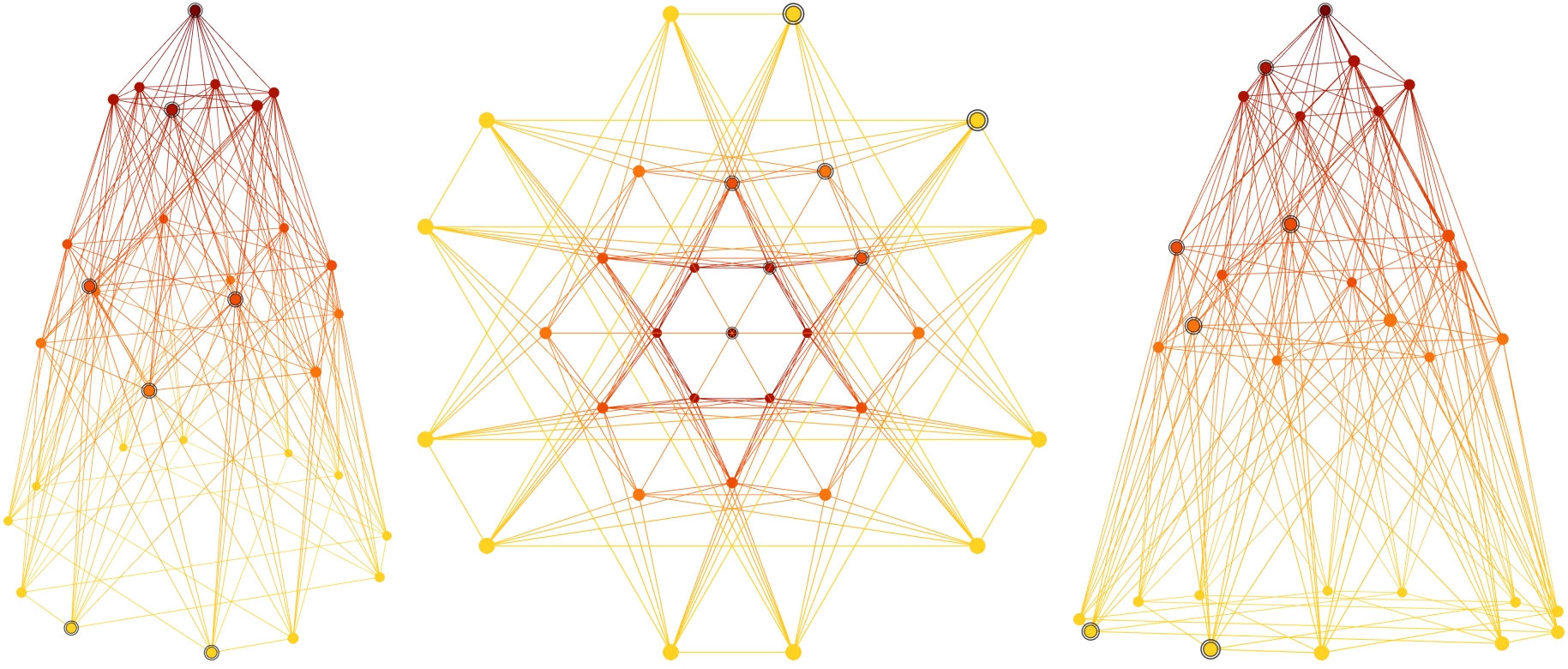


$\mathrm{Gr}_{\mathrm{SL}_2}$ (Harada–Henriques–Holm 2005)



$\Omega\mathrm{SU}(3)$ (Atiyah–Pressley 1983)

The GKM graph of $\mathrm{Gr}_{\mathrm{SL}_3}$ in three dimensions



Vertices: the T-fixed points λ, lifted to the paraboloid $E = |\lambda|^2$ and coloured by E.
`https://hausel.ista.ac.at/~thausel/zoo/gkm-k/`

Pólya on the affine Grassmannian

- the *tautological virtual class* at $L \subset \mathbb{C}((u))^n$ ($N \gg 0$):

$$\mathcal{Q} := \left[\mathbb{C}[[u]]^n / u^N \mathbb{C}[[u]]^n\right] - \left[L / u^N \mathbb{C}[[u]]^n\right] \in K_H(\mathrm{Gr}),$$

$$\mathcal{Q}|_{L_\lambda} = \sum_i z_i \, \frac{q^{2\lambda_i} - 1}{q^2 - 1}$$

- $E_m := \sum_{m_1+\cdots+m_n=m} \prod_{i=1}^{n} \prod_{j=0}^{m_i-1} \frac{z_i q^{2j} - Y_i}{1 - q^{2j+2}} \in$

$$\mathbb{Q}(q,z)[Y_1,\ldots,Y_n]^{S_n}\colon\ E_m(z_1 q^{2\lambda_1},\ldots,z_n q^{2\lambda_n}) = [\Lambda^m \mathcal{Q}]_{L_\lambda}$$

- $n = 1$: $E_m(z q^{2\lambda}) = z^m q^{m(m-1)} \begin{bmatrix}\lambda\\ m\end{bmatrix}_{q^2}$: *$q$-Pólya basis is* $\Lambda^m(\mathcal{Q})$

Theorem (Hausel–Löwit–Ngo, 2026)

$$\left\{ f \in \mathbb{Q}(q,z)[Y_1^{\pm 1},\ldots,Y_n^{\pm 1}]^{S_n} : \right.$$
$$\left. f(z_1 q^{2L_1},\ldots,z_n q^{2L_n}) \in \mathbb{Z}[q^{\pm 1}, z^{\pm 1}] \ \ \forall L \in \mathbb{Z}^n \right\}$$

$$= \bigoplus_{r \geq 0,\ \ell(\mu) < n} \mathbb{Z}[q^{\pm 1}, z^{\pm 1}]\, \delta_r(\Delta)\, E_{\mu_1} \cdots E_{\mu_{n-1}}, \qquad \Delta := \frac{Y_1 \cdots Y_n}{z_1 \cdots z_n}$$

The hybrid family quantum group $\mathbb{U}$

- $\mathbb{A} := \mathbb{Z}[q^{\pm 1}] \subset \mathbb{F} := \mathbb{Q}(q)$; $\mathrm{U}_{\mathbb{A}}^{\geq 0}$ the De Concini–Procesi (1993) integral form of the positive Borel of $\mathrm{U}_q(\mathfrak{gl}_n)$: generated by K_λ^+ ($\lambda \in \mathbb{Z}^n$) and $e_{ab} := (q - q^{-1})E_{ab}$
- $\mathrm{U}_{\mathbb{A}}^{\leq 0}$ its type-1 restricted dual: Lusztig's (1993) divided powers $F_i^{(r)}$, the toral units K_a^- and their Gaussian Newton binomials
- *Drinfeld double* $\mathbb{U} := \mathrm{U}_{\mathbb{A}}^{\geq 0} \otimes_{\mathbb{A}} \mathrm{U}_{\mathbb{A}}^{\leq 0}$, an $\mathbb{A}$-Hopf algebra:

$$[e_i, F_j] = \delta_{ij}\big(K_{\alpha_i}^+ - K_{\alpha_i}^-\big), \qquad [K_\lambda^+, K_\mu^-] = 0$$

  hybrid family quantum group (Hausel–Löwit, 2026)

- $t_a := K_a^+ K_a^-$ central group-like: the *family torus*; at $t_a = 1 \sim$ hybrid quantum groups of Habiro–Lê (2016) and Situ (2022); $\mathbb{U}_{\mathbb{F}}$: Berenstein–Greenstein (2017), Lacabanne (2018)

Verma characterisation and Harish–Chandra theorem

- *Family Verma module*

  $\mathbb{M}_{\mathbb{F}}(\lambda) := \mathbb{U}_{\mathbb{F}} \otimes_{\mathrm{U}_{\mathbb{F}}^{\geq 0}[(K_a^-)^{\pm 1}]} \mathbb{Q}(q)[t_a^{\pm 1}]\, m_\lambda,\ e_i m_\lambda = 0,$
  $K_a^- m_\lambda = q^{-\lambda_a} m_\lambda,\ K_a^+ m_\lambda = t_a q^{\lambda_a} m_\lambda;$
  lattice $\mathbb{M}(\lambda) := \mathbb{A}_t \mathrm{U}_{\mathbb{A}}^{\leq 0} m_\lambda,\ \mathbb{A}_t := \mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$

- $\mathrm{HC} : \mathbb{U}_{\mathbb{F}} \to \mathbb{U}_{\mathbb{F}}^0 = \mathbb{F}[(K_a^{\pm})^{\pm 1}]$ root-free PBW projection;
  $Y_a := q^{2(n-a)} K_a^+ (K_a^-)^{-1},\ D := K_1^- \cdots K_n^-$; on m_λ:
  $\mathrm{ev}_\lambda(Y_a) = t_a q^{2(\lambda_a + n - a)},\ \mathrm{ev}_\lambda(D) = q^{-\sum_a \lambda_a};$
  $D^2 \prod_a Y_a = q^{n(n-1)} \prod_a t_a.$

Theorem (Hausel–Löwit, 2026)

1. $\mathbb{U} = \left\{ x \in \mathbb{U}_{\mathbb{F}} :\ x\mathbb{M}(\lambda) \subseteq \mathbb{M}(\lambda)\ \ \forall \lambda \in \mathbb{Z}^n \right\}.$
2. $\mathrm{HC} :\ Z(\mathbb{U}) \xrightarrow{\sim}$
   $\left\{ f \in \mathbb{F}[t^{\pm 1}, D^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]^{S_n} :\ \mathrm{ev}_\lambda(f) \in \mathbb{A}_t\ \ \forall \lambda \in \mathbb{Z}^n \right\}$

Completed centre is equivariant K-theory

- $\chi_\lambda := \mathrm{ev}_\lambda \circ \mathrm{HC} : Z(\mathbb{U}) \to \mathbb{A}_t$, the scalar by which $Z(\mathbb{U})$ acts on $\mathbb{M}(\lambda)$. finite *Verma windows* $\Omega \subset \mathbb{Z}^n$, directed by inclusion:

$$J_\Omega := \bigcap\nolimits_{\lambda\in\Omega} \mathrm{Ann}_{\mathbb{U}}\mathbb{M}(\lambda), \quad I_\Omega := Z(\mathbb{U}) \cap J_\Omega = \bigcap\nolimits_{\lambda\in\Omega} \ker \chi_\lambda;$$

$$\widehat{\mathbb{U}} := \varprojlim\nolimits_\Omega \mathbb{U}/J_\Omega, \qquad \widehat{Z}(\mathbb{U}) := \varprojlim\nolimits_\Omega Z(\mathbb{U})/I_\Omega.$$

Theorem (Hausel–Löwit, 2026)

Fixed-point restriction gives, with $z_a = q^{2(n-a)}t_a$,

$$\widehat{Z}(\mathbb{U}) \xrightarrow{\sim} Z(\widehat{\mathbb{U}}) \xrightarrow[\sim]{\Theta} K_H(\mathrm{Gr}), \qquad i_\lambda^*\,\Theta(C) = \chi_\lambda(C).$$

- $B_\ell := \Theta^{-1}(b_{1,\ell}) \in Z(\mathbb{U})$ the *Bezrukavnikov classes*; by (Löwit, 2026) $Z(\mathbb{U}_\mathbb{F}) = \mathbb{F}[t_a^{\pm 1}][D^{\pm 1}, B_\ell]_{\ell\in\mathbb{Z}_{>0}}$.
- Their *elementary counterparts* by Newton identities: $E_0 = 1$, $E_m = \frac{1}{m}\sum_{\ell=1}^{m}(-1)^{\ell-1}B_\ell E_{m-\ell}$ are actually in $Z(\mathbb{U})$: *elementary Bezrukavnikov classes*
- $\Theta(E_m) = [\Lambda^m \mathcal{Q}]$: $i_\lambda^*\Theta(E_m) = E_m(z_1 q^{2\lambda_1}, \ldots, z_n q^{2\lambda_n})$, the q-Pólya generators E_m from before

Hecke operators on the affine Grassmannian

- $\mathcal{B} := L/uL$ the *universal bundle*; operators on $K_H(\mathrm{Gr})[\tau^{\pm 1}]$:
  - *Wilson lines* W_a: multiplication by $[\Lambda^a \mathcal{B}]$
  - *Hecke operators*: $\mathsf{F}_a(\tau) f := (\pi_a)_! \big(\lambda_{-\tau}(\Omega_{\pi_a})\, b_a^* f\big)$, $Z_a = \{uL \subseteq L' \subseteq L,\ \dim L/L' = a\}$, $\pi_a(L, L') = L$, $b_a(L, L') = L'$
- $K_H(\mathrm{Gr})[\tau^{\pm 1}] \hookrightarrow \mathcal{F} := \mathrm{Map}\big(\mathbb{Z}^n, \mathbb{Z}[q^{\pm 1}, z^{\pm 1}, \tau^{\pm 1}]\big)$;
  $(\mathsf{T}_I f)(\lambda) := f(\lambda + \varepsilon_I)$, $\varepsilon_I := \sum_{i \in I} \varepsilon_i$;
  $w_i(\lambda) := z_i^{-1} q^{-2\lambda_i}$ the roots of $\mathcal{B}^*$ at L_λ, $p := q^{-2}$:

  $$\mathsf{F}_a(\tau) = \sum\nolimits_{|I|=a} \prod\nolimits_{i \in I,\ j \notin I} \frac{\tau w_i - w_j}{w_i - w_j}\, \mathsf{T}_I \quad \text{on } K_H(\mathrm{Gr})[\tau^{\pm 1}],$$

  Macdonald's difference operators,
  with eigenclasses the Macdonald polynomials $P_\lambda(\mathcal{B}^*; p, \tau)$
- specialisations and limits: Schur ($\tau = p$), Hall–Littlewood ($p = 0$; spherical functions of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ at $\tau = \ell^{-1}$), q-Whittaker ($\tau = 0$), Jack ($p = \tau^\alpha$, $\tau \to 1$)
- Haiman (2001): Macdonald positivity, $\widetilde{K}_{\lambda\mu}(p, \tau) \in \mathbb{N}[p, \tau]$ ($n!$ theorem)

Translation is Hecke, and Macdonald

- $\mathcal{W}_k := S^{(k-1)n}V \otimes \det^{1-k}$, $k \geq 1$, with $\dim(\mathcal{W}_k)_0 = 1$
- generic $\Psi_\lambda : \mathbb{M}(\lambda) \to \mathbb{M}(\lambda) \otimes \mathcal{W}_k^*$ (Etingof–Varchenko 2000) with top component the zero-weight vector, suitably normalised
- for $z \in \widehat{Z}(\mathbb{U})$, $X_a^{(k)} z \in \widehat{Z}(\mathbb{U})$ acts on $\mathbb{M}(\lambda)$ by c_λ, where (generic family parameters) $c_\lambda \Psi_\lambda : \mathbb{M}(\lambda) \to \mathbb{M}(\lambda) \otimes \mathcal{W}_k^*$ is the quantum trace over $\Lambda^a V$ (V the standard module) of

$$\mathbb{M}(\lambda) \otimes \Lambda^a V \xrightarrow{\oplus_I \Psi_{\lambda+\varepsilon_I} \circ z} \mathbb{M}(\lambda) \otimes \Lambda^a V \otimes \mathcal{W}_k^* \xrightarrow{\check{R}} \mathbb{M}(\lambda) \otimes \mathcal{W}_k^* \otimes \Lambda^a V$$

Theorem (Hausel–Mellit–Ngo, 2026)

1. *W_a, $\mathsf{F}_a(\tau)$, $\det(\mathcal{B})^{-1}$, $\mathsf{F}_n(\tau)^{-1}$ generate a faithful action of an integral form of Cherednik's spherical DAHA on $K_H(\mathrm{Gr})[\tau^{\pm 1}]$*
2. *$X_a^{(k)}$ is well defined, and $\Theta\, X_a^{(k)} = q^{k\,a(n-a)}\, \mathsf{F}_a(q^{-2k})\, \Theta$*
3. *the Macdonald classes $\Theta^{-1} P_\mu(\mathcal{B}^*; q^{-2}, q^{-2k})$ are joint eigenvectors of the $X_a^{(k)}$ (cf. Etingof–Kirillov 1994)*

What is proved, and how we know

result	status
centre for GL_1: bilateral q-Pólya	Lean kernel
centre for GL_2, Newton lattice	Lean kernel
centre for GL_n, Harish-Chandra form	Lean kernel
completed centre $\cong K_H(\mathrm{Gr})$	Lean kernel
multivariate q-Pólya basis	Lean kernel
spherical DAHA action	proved informally; checked GL_2–GL_4
translation $=$ Hecke	proved informally

The open sandbox



ink proves,
orange computes,

blue verifies

- folders in which you do mathematical research with AI agents
- **the baby sandbox**: two agents, one shared notebook and Julia, running in ten minutes
- **the open sandbox**: a referee for every result, a certificate for every computation, your computer algebra system, Lean, collabs with other mathematicians' sandboxes
- Linux, macOS and Open Math OS

`https://git.ista.ac.at/ai-for-mathematics/open-sandbox`

Modern Representation Theory and Algebraic Geometry



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- 22–26 March: Geometry and Quantum Integrable Systems
- 12–16 April: Moduli Spaces of Sheaves, Enumerative Geometry, and Geometric Representation Theory
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